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CALCULUS OF VARIATIONS: FURTHER DISCUSSION OF THE FIRST VARIATION AND THE SOLUTION OF THE DIFFERENTIAL EQUATION $G = 0$ FOR SEVERAL INTERESTING EXAMPLES.

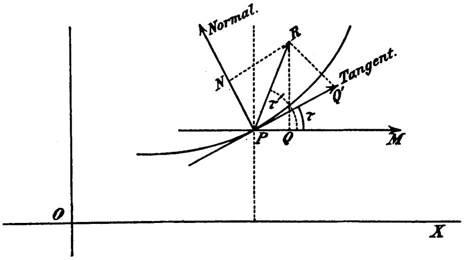
By Dr. Harris Hancock, Chicago, Illinois.

We saw ("The Annals," Vol. XI, p. 28) that

$$\delta I = \int_{t_0}^{t_1} G\,(y'\xi - x'\eta)\,dt + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t_0}^{t_1}, \qquad (1)$$

and we wish now to interpret $G\,(y'\xi - x'\eta)\,dt$.

Consider any point P of the curve, and suppose it is subjected to a sliding in the direction PR; also suppose that ds has the same sign as dt, so that when we write



$$PR = v\,,$$

$$PQ' = v_1\,,$$

$$PW = w\,,$$

$$\angle\ RPM = \tau'\,,$$

$$\angle\ QPM = \tau\,,$$

then is

$$x' = \frac{dx}{dt} = \frac{dx}{ds}\cdot\frac{ds}{dt} = \frac{ds}{dt}\cos\tau$$

and

$$y' = \frac{dy}{dt} = \frac{dy}{ds}\cdot\frac{ds}{dt} = \frac{ds}{dt}\sin\tau\,.$$

Substitute these values in (1) and we have :

$$\delta I = \int_{t_0}^{t_1} G\,(\xi \sin\tau - \eta \cos\tau)\; ds + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t_0}^{t_1}. \tag{2}$$

To denote that a sliding takes place along the line PR we must make (see p. 25) the substitution :

$$\begin{array}{c||l} x & x + e\xi\,, \\ y & y + e\eta\,, \end{array}$$

where e is the variable parameter.

Denote the whole sliding that takes place along PR by v, then the component in the x direction, that is ξ, is

$$\xi = v \cos \tau'\,,$$

and

$$\eta = v \sin \tau'\,.$$

Further decomposing v into two components in the direction of the tangent and the normal to the curve at P, and denoting these components respectively by v_1 and w, we have

$$v_1 = v \cos(\tau' - \tau)\,,$$

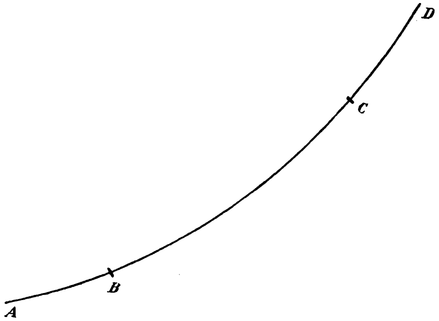
$$w = v \sin(\tau' - \tau) = v \sin\tau' \cos\tau - v \cos\tau' \sin\tau$$

$$= \eta \cos\tau - \xi \sin\tau\,.$$

Hence from (2)

$$\delta I = -\int_{t_0}^{t_1} Gw\,.\,ds + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\cdot\;\right]_{t_0}^{t_1},$$

from which is seen *that only that component of the sliding enters under the sign of integration which is in the direction of the normal.*



It is shown below that all slidings in the direction of the tangent can give only such terms that are completely integrable.

When a curve slides along into itself, this means nothing else than that we traverse the curve by referring the points in another manner to an independent variable. A sliding of the curve into itself is clearly nothing else than a sliding of the single points in the directions of their tangents; and we will now show that this sliding gives only such terms of the first variation which are free from the sign of integration. If we make this slide, then the integral I has either in all the elements the same quantities, or it happens that at the beginning a portion of the curve is cut off and at the end a portion enters in addition, so that at the beginning elements are lost, and at the end additional elements enter.

By means of the formulæ below we will prove in a direct manner that *the variation in the direction of the tangent brings forth only such terms for the first variation that are free from the sign of integration.*

Writing (see "The Annals," p. 26)

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial x'} \xi' + \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right] dt,$$

and noting that $\tau = \tau'$ when the sliding is in the direction of the tangent, we have

$$\xi = v \frac{dx}{ds} = v \cos \tau',$$

and

$$\eta = v \frac{dy}{ds} = v \sin \tau',$$

and hence

$$\delta I = \int_{t_0}^{t_1} \left\{ v \left[\frac{\partial F}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial F}{\partial y} \cdot \frac{dy}{ds} \right] + \frac{\partial F}{\partial x'} \cdot \frac{d}{dt} \left[v \frac{dx}{ds} \right] + \frac{\partial F}{\partial y'} \cdot \frac{d}{dt} \left[v \frac{dy}{ds} \right] \right\} dt.$$

But

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'} \cdot \frac{d}{dt} \left[v \frac{dx}{ds} \right] = \left[\frac{\partial F}{\partial x'} \cdot v \frac{dx}{ds} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} v \frac{dx}{ds} \cdot \frac{d}{dt} \left[\frac{\partial F}{\partial x'} \right] dt;$$

so that

$$\begin{aligned} \delta I &= \int_{t_0}^{t_1} \left\{ v \frac{dx}{ds} \left[\frac{\partial F}{\partial x} - \frac{d}{dt} \left[\frac{\partial F}{\partial x'} \right] \right] + v \frac{dy}{ds} \left[\frac{\partial F}{\partial y} - \frac{d}{dt} \left[\frac{\partial F}{\partial y'} \right] \right] \right\} dt \\ &\qquad + \left[\frac{\partial F}{\partial x'} \cdot v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} G (y'\xi - x'\eta) \, dt + \left[\frac{\partial F}{\partial x'} v \frac{dx}{ds} + \frac{\partial F}{\partial y'} v \frac{dy}{ds} \right]_{t_0}^{t_1}. \end{aligned}$$

But

$$(y'\xi - x'\eta) = \left[y'v\frac{dx}{ds} - x'v\frac{dy}{ds}\right] = \frac{v}{ds}\left[y'\frac{dx}{dt} - x'\frac{dy}{dt}\right]dt = 0\,,$$

so that everything under the sign of integration drops out, leaving

$$\delta I = \left[\frac{\partial F}{\partial x'}\,v\frac{dx}{ds} + \frac{\partial F}{\partial y'}\,v\frac{dy}{ds}\right]_{t_0}^{t_1}$$

Hence if we have any sliding, and resolve this sliding into two components, of which the one is parallel to the direction of the tangent, and the other is parallel to the direction of the normal, then the result of the sliding in the direction of the tangent is seen only in the terms which have reference to the limits, and all these terms are complete differentials under the sign of integration; whereas the sliding in the direction of the normal is seen only under the sign of integration in the first variation.

Note.—This can be generalized to the case where we deal with surfaces; i. e., where we have to do with double integrals.

INTEGRATION OF THE DIFFERENTIAL EQUATION $G = 0$ FOR THE FIRST THREE PROBLEMS.*

Problem I. Problem of the surface of rotation which is to have the least surface area.

We have already derived the integral (see "The Annals," Vol. IX, p. 183)

$$S = 2\pi\int_{t_1}^{t_1} y\,\sqrt{x'^2 + y'^2}\,dt\,;$$

so that here

$$F = y\sqrt{x'^2 + y'^2}\,;$$

and consequently

$$\frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{x'^2 + y'^2}}\,,\quad \frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{x'^2 + y'^2}}\,.$$

From this follows, that

$$\frac{\partial F}{\partial x'} \text{ and } \frac{\partial F}{\partial y'}$$

are proportional to the direction cosines of the tangent at any point $x(t)$, $y(t)$; and since, as shown above, Vol. XI, p. 31, these quantities vary everywhere in a continuous manner, it follows also that the direction of the curve varies everywhere in a continuous manner except for the case where y can be $= 0$.

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* These problems were stated by Prof. Weierstrass in his lectures (cf. the "Annals," Vol. IX, p. 179), and the solutions that follow are essentially due to him.

In this last case the curve consists of several regular traces; *in general*, however, of only one.

Since x is not contained explicitly in F, it is expedient to use $G_1 = 0$ instead of $G = 0$.

The differential equation for the curve that we are seeking is then

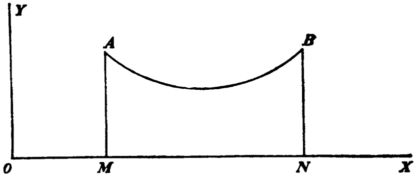
$$\frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{x'^2 + y'^2}} = C,$$

where C is an arbitrary constant.

If we take the arc s as independent variable instead of the variable t, the above equation may be written:

$$y\frac{dx}{ds} = C.$$

Suppose that $C = 0$, then C must retain this value also in the whole interval $t_0 \ldots t_1$, that is, along the curve AB; and since y is not zero for the point A, it follows that $\frac{dx}{ds} = \cos \alpha = 0$ (where α is the angle, which the tangent makes with the x-axis), $\cos \alpha$ must remain $= 0$ until $y = 0$; that is, the point which



describes the curve must move upon the ordinate A to the point M. At this point $\frac{dx}{ds}$ cannot and must not equal to zero if the point is to move to B. Hence at M there is a sudden change in the direction of the curve. The point then moves farther upon the x-axis, but must again leave this axis in order to go to B, which is only possible (when $C = 0$) by having $\frac{dx}{ds} = 0$, and then the point moves on the ordinate of B. Hence for the case $C = 0$, the curve, which satisfies the differential equation, consists of the broken line $AMNB$.

But if $C \neq 0$, then $C \neq 0$ throughout the whole extent of the curve. Since $y \neq 0$ at the beginning, it follows that $\frac{dx}{ds}$ is finite and $\neq 0$ and varies throughout the whole extent of the curve in a continuous manner.

Hence y must also vary everywhere in a continuous manner and can never be zero, that is, the curve consists of one single regular trace, which lies wholly on the positive side of the x-axis.

From the relation

$$y\frac{dx}{ds} = C$$

we have

$$y^2dx^2 = C^2(dx^2 + dy^2)$$

a differential equation which has been treated at length in "The Annals," Vol. X, p. 84 and p. 159.

Problem II. Problem of the brachistocrone (cf. "The Annals," Vol. IX, p. 183).

The time of falling of the movable point expressed n terms of $x(t)$ and $y(t)$ from the point A to the point B is

$$T = \int_{t_0}^{t_1} \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + a^2}}\, dt\,.$$

In order that this integral should in reality express the time of falling, since within the integral the time and therefore also the increment dt is an essentially positive quantity, the two roots must always have the same sign. And since $\sqrt{4gy + a^2}$ can always be chosen positive, it follows that $\sqrt{x'^2 + y'^2}$ must within the interval $t_0 \ldots t_1$ be positive.

It might happen, however, if we express x and y in terms of t, that x' and y' might both vanish for a value of t within the interval $t_0 \ldots t_1$. In this case the curve has at the point x, y, which belongs to this value of t, a singular point, at which the velocity of the moving point is zero.

Suppose that this is the case for $t = t'$ and that the corresponding point is x_0, y_0, so that we have

$$x = x_0 + A\,(t - t')^m + \ldots,$$
$$y = y_0 + B(t - t')^m + \ldots,$$

where $m \geqq 2$, and at least one of the two quantities A and $B \neq 0$.

Then is:

$$x'^2 + y'^2 = m^2(A^2 + B^2)(t - t')^{2(m-1)} + \ldots,$$
$$\sqrt{x_1^2 + y'^2} = m\sqrt{A^2 + B^2}(t - t')^{m-1} + \ldots.$$

And here we may suppose $\sqrt{A^2 + B^2}$ positive.

If now m is odd, then for small values of $t - t'$, the expression on the right is positive, and hence $\sqrt{x'^2 + y'^2}$ has always the positive sign.

If on the contrary m is even, $=$ say 2, then in this case the curve has at the point x_0, y_0 a cusp, since here $\sqrt{x'^2 + y'^2}$ has a positive or a negative value according as $t > t'$ or $t < t'$.

If therefore the above integral is to express the time, $\sqrt{x^2 + y^2}$ cannot always be put equal to the same series of t, but must after passing the *cusp* be put equal to the opposite value of the series. We therefore limit ourselves to the consideration of a portion of the curve which is free from singular points.

Such limitations must be often made in problems, since otherwise the integrals have no definite meaning.

Hence with this supposition $\sqrt{x'^2 + y'^2}$ will never $= 0$.

We may then write:

$$F = \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + a^2}}, \tag{2}$$

and consequently:

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{4gy + a^2}} \cdot \frac{x'}{\sqrt{x'^2 + y'^2}},$$

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{4gy + a^2}} \cdot \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

And from this we conclude in a similar manner as in the first example, that $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$, are proportional to the direction cosines of the tangent of the curve at the point x, y. Since now $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$ vary in a continuous manner along the the whole curve, and since further $\sqrt{4gy + a^2}$ has a definite value which is different from zero, it follows also that the direction of the required curve varies in a continuous manner, or the curve must consist of one single trace.

Also here F is independent of x, and consequently we employ the differential equation $G_1 = 0$; from which we have

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{4gy + a^2}} \cdot \frac{x'}{\sqrt{x'^2 + y'^2}} = c, \tag{3}$$

where c is an arbitrary constant.

If c was equal to 0, then in the whole extent of the curve c must $= 0$; and consequently, since $\sqrt{4gy + a^2}$ is neither 0 nor ∞, $\frac{x'}{\sqrt{x'^2 + y'^2}} = \cos a$ must always $= 0$, that is, the curve must be a vertical line. And neglecting this self-evident case, c must have a definite value which is always the same for the whole curve and different from zero.

From (3) follows that:

$$dx^2 = c^2(4gy + \alpha^2)(dx^2 + dy^2)\,.$$

Or, if we absorb $4g$ in the arbitrary constant and write

$$\frac{\alpha^2}{4g} = a\,,$$

we have

$$dx^2 = c^2(y + a)(dx^2 + dy^2)\,;$$

whence

$$dx = \frac{c(y + a)\,dy}{\sqrt{(y + a)(1 - c^2(y + a))}}\,.$$

In order to perform this last integration write

$$du = \frac{c\,dy}{\sqrt{(y + a)(1 - c^2(y + a))}}\,; \tag{5}$$

therefore

$$dx = (y + a)\,du\,.$$

In the expression for du, write

$$2c^2(y + a) = 1 - \xi\,.$$

Then is

$$2(1 - c^2(y + a)) = 1 + \xi\,,$$

and

$$2c^2dy = -\,d\xi\,.$$

Therefore

$$du = -\frac{d\xi}{\sqrt{1 - \xi^2}}\,,$$

and hence

$$\xi = \cos u\,.$$

Here the constant of integration may be omitted, since u itself is fully arbitrary.

Hence

$$\left.\begin{aligned} y + a &= \frac{1}{2c^2}(1 - \cos u)\,, \\ x + x_0 &= \frac{1}{2c^2}(u - \sin u)\,; \end{aligned}\right\} \tag{8}$$

and from (6)

and these equations represent a cycloid.

The constants of integration x_0, c are determined from the condition that the curve is to go through the two points A and B. Now develop x and y in powers of u, then in y the lowest power is u^2, and in x it is u^3. So that the curve has in reality a cusp for $u = 0$, and this is repeated for $u = 2\pi, 4\pi, \ldots$.

A and B must lie within such an interval of a (for example within the interval $2n\pi$ and $2(n+1)\pi$), that is, between two consecutive cusps.

The curve may be constructed if we draw a horizontal line through the point $-x_0, -a$, and construct on the under side of this line a circle with radius $\frac{1}{2c^2}$, which touches the horizontal line at the point $-x_0, -a$. Let this circle roll in the positive x direction on the horizontal lines, then the original point of contact describes a cycloid which goes through A and B and which satisfies the differential equation.

Problem III. Problem of the shortest line on a surface.

This problem cannot in general be solved, since the variables in the differential equation cannot be separated, and consequently the integration of this equation cannot be performed.

Only in a few cases has one succeeded in carrying through this integration, and in this manner of representing the curve which satisfies the differential equation through closed expressions.

This is for example the case with the plane, the sphere, and with all other surfaces of the second degree.

As a simple example we will take the problem of the shortest line between two points on the surface of a sphere. The radius of the sphere is put $= 1$, and the equation of the sphere is given in the form:

$$x^2 + y^2 + z^2 = 1 .$$

Now writing:

$$\left.\begin{aligned} x &= \cos u , \\ y &= \sin u \cos v , \\ z &= \sin u \sin v ; \end{aligned}\right\} \qquad (1)$$

then $u = \text{const.}$ and $v = \text{const.}$ are the equations of the parallel circles and of the meridians respectively.

The element of arc is

$$ds = \sqrt{du^2 + \sin^2 u dv^2} ; \qquad (2)$$

and consequently the integral which is to be made a minimum is:

$$L = \int_{t_0}^{t_1} \sqrt{u'^2 + v'^2 \sin^2 u}\, dt ; \qquad (3)$$

so that here we have

$$F = \sqrt{u'^2 + v'^2 \sin^2 u}\,, \tag{4}$$

and

$$\left.\begin{aligned} \frac{\partial F}{\partial u'} &= \frac{u'}{\sqrt{u'^2 + v'^2 \sin^2 u}}\,, \\ \frac{\partial F}{\partial v'} &= \frac{v' \sin^2 u}{\sqrt{u'^2 + v'^2 \sin^2 u}}\,, \end{aligned}\right\} \tag{5}$$

Since F does not contain the quantity v, we will use the equation $G_1 = 0$ and have:

$$\frac{\partial F}{\partial v'} = \frac{v' \sin^2 u}{\sqrt{u'^2 + v'^2 \sin^2 u}} = c\,,$$

where c is an arbitrary constant which has the same value along the whole curve.

If for the initial point A of the curve $u \neq 0$, and consequently, therefore, not the north pole of the sphere, then c will only then be everywhere $= 0$, if $v' = 0$, and when, therefore, $v = \text{const}$. And then A and B must lie on the same meridian, and this is a solution of the problem.

If now this is not the case, then always $c \neq 0$, and as is easy to see $c < 1$; we can therefore write $\sin c$ instead of c, and have:

$$\frac{v' \sin^2 u}{\sqrt{u'^2 + \sin^2 u v'^2}} = \sin c\,,$$

or

$$dv = \frac{\sin c\, du}{\sin u \sqrt{\sin^2 u - \sin^2 c}}\,. \tag{7}$$

If we write

$$\cos a = \cos c \cos w\,, \tag{8}$$

then is

$$dv = \frac{\sin c\, dw}{1 - \cos^2 c \cos^2 w}\,;$$

and when integrated this gives:

$$tg\,(v - \beta) = \frac{1}{\sin c}\, tg w\,, \tag{9}$$

where β represents an arbitrary constant.

Eliminating w by means of (8), we have:

$$tg u \cos (v - \beta) = tg c\,. \tag{10}$$

This is the equation of the curve which we are seeking expressed in the spherical coordinates u, v.

In order to study their meaning closer, we may express u, v separately through the arc s, where s is measured from the intersection of the zero meridian with the shortest line.

Through (7) the expression (2) goes into:

$$\delta s = \frac{\sin u \, du}{\sqrt{\sin^2 u - \sin^2 c}},$$

and this owing to the substitution (8) becomes

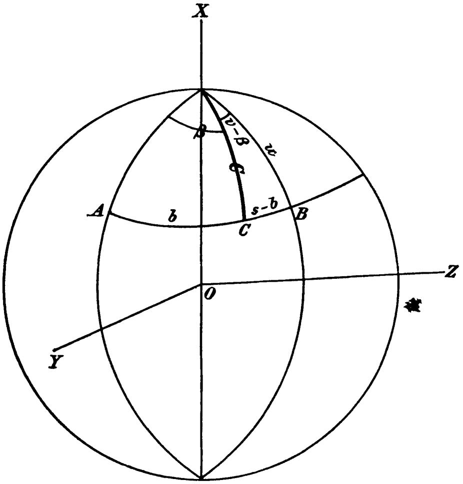
$$\delta s = \delta w ,$$

and, therefore, if b is a new constant,

$$s - b = w . \tag{11}$$

Hence from equations (8) and (9) we have the following equations:

$$\left.\begin{aligned} \cos u &= \cos c \cos (s - b) , \\ ctg (v - \beta) &= \sin c \; ctg (s - b) . \end{aligned}\right\} \tag{12}$$



But these are relations which exist between the sides and the angles of a right angle spherical triangle.

If we consider that meridian drawn from the north pole, which cuts at right angles the curve we are seeking, then this meridian forms with the curve and any other meridian a triangle, to which the above relations may be applied.

Therefore the curve which satisfies the differential equation must itself be the arc of a great circle. The constants of integration c, b, β are determined from the conditions that the curve is to pass through the two points A and B.

The geometrical interpretation is that c is the length of the geodetic normal from the point $u = 0$ to the shortest line; $s - b$, the arc from the foot of this normal to any point of the curve, and $v - \beta$ the angle opposite this arc, that is, the difference of length between the end points of this arc.

If we therefore assume that the zero meridian passes through A, and if, accordingly, we measure the arc from A, then b is the length of arc of the shortest line from A to the normal, and β the geographical longitude of the foot of this normal.